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**APPENDIX A**

**CONVERGENCE OF ADMM ON A QUADRATIC PROBLEM**

We consider the problem

\[
\begin{align*}
\min_{x,z} & \quad f(x) + g(z) \\
\text{s.t.} & \quad Ax + Bz = c.
\end{align*}
\]

(1)

where \( f \) and \( g \) are quadratic functions with positive definite Hessians \( H_f \) and \( H_g \) respectively. Without loss of generality, we may translate \( x \) and \( z \) so that the minima of \( f \) and \( g \) occur at \( x = 0 \) and \( z = 0 \) respectively. Thus we have \( f(x) = \frac{1}{2}x^T H_f x \) and \( g(z) = \frac{1}{2}z^T H_g z \). We assume that the constant term \( c \) in the constraint is zero; a nonzero \( c \) does not affect the convergence rate, though of course it must lie in the image of \( A \) and \( B \). Finally, we rescale the variables via \( \bar{x} = Px \) and \( \bar{z} = Qz \) such that \( P^T P = H_f \) and \( Q^T Q = H_g \), and rescale the constraint by a matrix \( W \). This yields the equivalent problem

\[
\begin{align*}
\min_{\bar{x},\bar{z}} & \quad \frac{1}{2} \|\bar{x}\|^2 + \frac{1}{2} \|\bar{z}\|^2 \\
\text{s.t.} & \quad \frac{W A}{A} \bar{x} + \frac{W B}{B} \bar{z} = 0.
\end{align*}
\]

(2)

Note that ADMM applied this problem has exactly the same convergence as when applied to the original objective \( \min f(x) + g(x) \) with the rescaled constraint \( WAx + WBz = 0 \).

In the \( x \)-step of ADMM, \( \bar{x}^{n+1} \) is determined completely by \( \bar{z}^n \) and \( u^n \) via

\[
\bar{x}^{n+1} = \arg \min_{\bar{x}} \left( \frac{1}{2} \|\bar{x}\|^2 + \frac{\rho}{2} \|A\bar{x} + B\bar{z}^n + u^n\|^2 \right) = -(I + \rho A^T A)^{-1} A^T (B\bar{z}^n + u^n).
\]

(3)

Therefore, the progress of the ADMM iterations is determined by the \( z \)- and \( u \)-steps. After some algebra, we obtain

\[
\begin{align*}
\bar{z}^{n+1} &= SB^T (\bar{A} \bar{R}^T \bar{B} \bar{z}^n - (I - \bar{A} \bar{R}^T) u^n), \\
u^{n+1} &= (I - BSB^T) (\bar{A} \bar{R}^T \bar{B} \bar{z}^n + (I - \bar{A} \bar{R}^T) u^n).
\end{align*}
\]

(4)

\[ \text{and its convergence rate is determined by the spectral radius of the recurrence matrix,} \]

\[
r \left( \begin{bmatrix} SB^T \\ I - BSB^T \end{bmatrix} \begin{bmatrix} \bar{A} \bar{R}^T \bar{B} & I - \bar{A} \bar{R}^T \end{bmatrix} \right) = r \left( \begin{bmatrix} \bar{A} \bar{R}^T \bar{B} & I - \bar{A} \bar{R}^T \end{bmatrix} \begin{bmatrix} SB^T \\ I - BSB^T \end{bmatrix} \right) = r(\bar{A} \bar{R}^T BSB^T + (I - \bar{A} \bar{R}^T)(I - BSB^T)).
\]

(7)

In general, this expression cannot be simplified further. However, if \( \rho = 1 \) and \( B = I \), then we obtain \( S = (I + \rho B^T B)^{-1} = \frac{1}{2} I \), and the convergence rate becomes simply

\[
r \left( \frac{1}{2} \bar{A} \bar{R}^T + \frac{1}{2} (I - \bar{A} \bar{R}^T) \right) = \frac{1}{2}.
\]

(8)

This is achieved when \( B = WBQ^{-1} = I \), i.e., \( Q = WB \). Further, as \( Q \) is any matrix which satisfies \( Q^T Q = H_g \), we only require \( (WB)^T (WB) = H_g \), equivalently, \( \frac{1}{2} \|WBx\|^2 = g(x) \).

**APPENDIX B**

**PROOF THAT PROJECTIVE DYNAMICS \( \approx \) ADMM**

We apply ADMM to the projective dynamics energy

\[
U_i(z_i) = \min_{p_i \in C_i} k_i \|z_i - p_i\|^2.
\]

(9)

In our formulation of ADMM, we have one parameter \( W \). We define \( W \) via \( W_i = w_i I = \sqrt{k_i} I \), so that \( W^TW = K \).

Then the energy can be conveniently expressed in terms of a single constraint manifold, \( C = C_1 \times C_2 \times \cdots \times C_m \):

\[
U_s(z) = \min_{p \in C} \frac{1}{2} (z - p)^T K (z - p)
\]

(10)

\[
= \min_{p \in C} \frac{1}{2} \|W(z - p)\|^2.
\]

(11)
Now the \( z \)-step of ADMM becomes
\[
\begin{align*}
z^{n+1} &= \arg\min_z \left(U_n(z) + \frac{1}{2} \|W(Dx^{n+1} - z + \bar{u}^n)\|^2\right) \\
&= \arg\min_z \left(\min_{p \in C} \frac{1}{2} \|W(z - p)\|^2 + \frac{1}{2} \|W(z - y)\|^2\right)
\end{align*}
\] (12)
where \( y = Dx^{n+1} + \bar{u}^n \). Consider the underlying minimization
\[
\min_{z, p \in C} \frac{1}{2} \|W(z - p)\|^2 + \frac{1}{2} \|W(z - y)\|^2.
\] (14)
For any fixed \( p \in C \), the minimum is attained at \( z = \frac{1}{2} (p + y) \) and its value is \( \frac{1}{2} \|W(p - y)\|^2 \). Therefore, the optimal \( p \) must minimize \( \|W(p - y)\|^2 \). For our choice of \( W \) and \( C \) this amounts to minimizing \( u_i\|p_i - y_i\|^2 \) independently for each \( i \), that is, choosing \( p_i = \text{proj}_C y_i = \text{proj}_C (D_i x^{n+1} + \bar{u}^n) \). So in fact we have
\[
\begin{align*}
p^{n+1} &= \text{proj}_C (D_i x^{n+1} + \bar{u}^n), \\
z^{n+1} &= \frac{1}{2} (p^{n+1} + Dx^{n+1} + \bar{u}^n).
\end{align*}
\] (15, 16)
Armed with (15)–(16), we will now eliminate \( z \) from the ADMM update rules in favour of \( p \). The \( u \)-update becomes
\[
\begin{align*}
\bar{u}^{n+1} &= \bar{u}^n + Dx^{n+1} - z^{n+1} \\
&= \bar{u}^n + Dx^{n+1} - \frac{1}{2} (p^{n+1} + Dx^{n+1} + \bar{u}^n) \\
&= \frac{1}{2} (Dx^{n+1} + \bar{u}^n - p^{n+1}).
\end{align*}
\] (17, 18, 19)
Conveniently, this also means that after the \( \bar{u} \)-update,
\[
\begin{align*}
z^{n+1} - \bar{u}^{n+1} &= \frac{1}{2} (p^{n+1} + Dx^{n+1} + \bar{u}^n) \\
&= \frac{1}{2} (Dx^{n+1} + \bar{u}^n - p^{n+1}) \\
&= p^{n+1}.
\end{align*}
\] (20, 21)
The \( x \)-update is now
\[
\begin{align*}
x^{n+1} &= (M + \Delta^2 D^T W^T W D)^{-1} \\
&= (M + \Delta^2 D^T W^T W (x^n - \bar{u}^n))^{-1} \left(M \bar{x} + \Delta^2 D^T W^T W p^n \right),
\end{align*}
\] (22, 23)
which is almost exactly like the \( p \)-update in projective dynamics, except for the presence of the dual variables \( \bar{u}_i \).

Finally, the \( u \)-update remains
\[
\begin{align*}
\bar{u}_i^{n+1} &= \frac{1}{2} (D_i x^{n+1} + \bar{u}_i^n - p_i^{n+1}) \\
&= \frac{1}{2} (D_i x^{n+1} + \bar{u}_i^n - p_i^{n+1})
\end{align*}
\] (24, 25, 26, 27)
which has no counterpart in projective dynamics.

So far we have seen that for a general constraint manifolds \( C_i \), projective dynamics and ADMM are extremely similar, with the only difference being the presence of the \( \bar{u}_i \) variables and their corresponding update rules. In the special case when the constraints are linear, that is, the manifolds \( C_i \) are affine, we further show that the two algorithms become identical.

Let \( C_i \) be an affine subspace with normal space \( N_i \). Then the projection operator \( \text{proj}_{C_i} \) has the properties that
\[
\begin{align*}
z_i &= \text{proj}_{C_i} z_i \in N_i, \\
\forall n \in N_i : \text{proj}_{C_i} (z_i + n) &= \text{proj}_{C_i} z_i.
\end{align*}
\] (28, 29)
We can see that
\[
\begin{align*}
\bar{u}_i^{n+1} &= \frac{1}{2} \left(D_i x^{n+1} + \bar{u}_i^n - p_i^{n+1}\right) \\
&= \frac{1}{2} \left(D_i x^{n+1} + \bar{u}_i^n - \text{proj}_{C_i} (D_i x^{n+1} + \bar{u}_i^n)\right) \\
&\in N_i,
\end{align*}
\] (30, 31, 32)
and so
\[
\begin{align*}
p_i^{n+1} &= \text{proj}_{C_i} (D_i x^{n+1} + \bar{u}_i^n) \\
&= \text{proj}_{C_i} D_i x^{n+1}
\end{align*}
\] (33, 34)
as long as \( u_i^0 \in N_i \) (for example, if we initialize \( u_i^0 = 0 \)).

This proves the equivalence of projective dynamics and ADMM for linear constraints. Furthermore, nonlinear constraints that are smooth can be well approximated by a linearization in the neighborhood of the solution, so both algorithms should behave similarly as they approach convergence.